# Calculus I

Hussein Elguindi
helguind@uwaterloo.ca
elguindi.xyz

December 6, 2022

This document started as an accumulating note over the course of high school AP Calculus. However, it has evolved into my university Calculus I reference notes. The content of this document is loosely based on the University of Waterloo's MATH 117 course. It is intended to be a condensed, yet thorough, combination of every source I used to learn the material.

For any suggestions, feedback, or comments, please email me at the above-stated address.

# Contents



# Definitions

## Theorems

# 1 Functions

**Definition 1.1** (Functions)**.** A function from a set $A$ to a set $B$, for example $f\colon A \to B$, is a rule/map which assigns elements in $B$ to each element in $A$.

## 1.1 Properties

**Definition 1.2** (Injective Functions)**.** For an injective function $f\colon A \to B$, $\forall\, a, b \in A$, if $f(a) = f(b)$, then $a = b$.

**Note.** An injective function is known as being one-to-one.

**Note.** If a function is injective, it will pass the Horizontal Line Test. If a function is not injective, its domain may be restricted so that the restriction of the function is injective and, therefore, invertible (assuming it is also surjective).

**Definition 1.3** (Surjective Functions)**.** For a surjective function $f\colon A \to B$, $\forall\, y \in B$, $\exists\, x \in A$ such that $f(x) = y$.

**Note.** In other words, every element from the codomain $B$ is mapped to (from at least one input in the domain $A$).

**Definition 1.4** (Bijective Functions)**.** A function is bijective if and only if it is injective and subjective.

## 1.2 Composition

**Definition 1.5** (Function Composition)**.** The composite function $h$ is the result of the composition of function $g$ in function $f$:

$$h(x) = f(g(x)) = (f \circ g)(x).$$

**Note.** Suppose the function $g$ is composed in function $f$, then the domain of the composite function $f(g(x)) = (f \circ g)(x)$ is all $x$ in the domain of $g$ such that $g(x)$ is in the domain of $f$.

## 1.3 Inverses

**Definition 1.6** (Inverse Functions)**.** A function $g\colon B \to A$ is said to be the inverse of the function $f\colon A \to B$ if

$$\forall\, x \in A,\ g(f(x)) = x.$$

**Definition 1.7** (Invertible Functions). A function is said to be invertible if it is bijective.

**Note.** Inverses are unique.

**Theorem 1.1** (Solving for Inverses). The inverse of a function $f$ is found by solving for $x$ in the relation $y = f(x)$.

## 1.4 Piecewise Defined Functions

**Definition 1.8** (Absolute Value Function). The absolute value function is a piecewise-defined function as

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}.$$

**Definition 1.9** (Heaviside Function (Unit Step Function)). The Heaviside function is a piecewise-defined function as

$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

**Example 1.1** (Heaviside Function).

$$f(x) \cdot H(x) = \begin{cases} f(x), & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

**Example 1.2.** Let $a \in \mathbb{R}$. Consider the function $H(x - a)$:

$$\begin{aligned} H(x-a) &= \begin{cases} 1, & x - a \geq 0 \\ 0, & x - a < 0 \end{cases} \\ &= \begin{cases} 1, & x \geq a \\ 0, & x < a \end{cases}. \end{aligned}$$

# 2 Continuity and Limits

**Definition 2.1** (End Behaviour Models). A function $g$ is said to be the end behaviour model of a function $f$ if $g(x)$ models the behaviour of $f(x)$ as $x \to \pm\infty$. Mathematically, $g$ is an end behaviour model for $f$ if and only if

$$\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = 1.$$

**Remark.** A function can have different left and right end behaviour models.

## 2.1 Continuity

**Definition 2.2** (Point Continuity). A function $f$ is continuous at an interior point if and only if

$$\lim_{x \to c} f(x) = f(c).$$

A function $f$ is continuous at a left endpoint $a$ or a right endpoint $b$ of a closed interval if and only if

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively.}$$

A function is said to be continuous on an interval if and only if every point on that interval is continuous.

**Theorem 2.1** (Intermediate Value Theorem). If a function $f$ is continuous on the closed interval $[a, b]$ then it takes on every value between $f(a)$ and $f(b)$.

In other words, for all numbers $k$ between $f(a)$ and $f(b)$, there exists at least one point $c \in (a, b)$ such that $f(c) = k$.

## 2.2 Composite Functions

**Theorem 2.2** (Continuity of Composite Functions). If a function $g$ is continuous at $c$ and function $f$ is continuous at $g(c)$, then the composite function given by $f \circ g$ is continuous at $c$.

**Theorem 2.3** (Limits of Composite Functions)**.** The limit of composite function $f \circ g$ is

$$\lim_{x\to c}(f\circ g)(x) = \lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right),$$

if and only if

1. $\lim_{x\to c} g(x)$ exists, and
2. $f$ is continuous at $\lim_{x\to c} g(x)$.

## 2.3 Limit Evaluation Techniques

**Theorem 2.4** (The Squeeze Theorem)**.** Let $h$, $f$, and $g$ be real-valued functions. If $h(x) \le f(x) \le g(x)$ for all $x$ in an interval containing $c$, except possibly at $c$ itself, and if

$$\lim_{x\to c} h(x) = L = \lim_{x\to c} g(x),$$

then

$$\lim_{x\to c} f(x) = L.$$

**Theorem 2.5** (l'Hôpital's Rule)**.** Let $c$ be a real number. Let $f$ and $g$ be real-valued functions, differentiable on an interval except possibly at $c$. l'Hôpital's Rule states that, if

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0 \text{ or } \pm\infty,$$

then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

if the limit exists (or is $\pm\infty$).

**Remark.** A proof of this can be found in Appendix A.1.

If direct subsitution into a limit results in different indeterminate forms, they can sometimes be algebraicly manipulated into $0/0$ or $\infty/\infty$ so that l'Hôpital's rule can be applied.

**Definition 2.3** (Indeterminate Forms)**.** The indeterminate forms are:

$$\frac{0}{0},\quad \frac{\infty}{\infty},\quad 0\times\infty,\quad \infty-\infty,\quad 0^0,\quad 1^\infty,\quad \text{and}\quad \infty^0.$$

# 3 Derivatives

**Definition 3.1** (Limit Definition of the Derivative)**.** The derivative of a differentiable function $f$ at $x$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

or

$$f'(x) = \lim_{c \to x} \frac{f(c) - f(x)}{c - x}.$$

**Remark.** A function can have different left and right derivatives at a point (e.g. $f(x) = |x|$ at $x = 0$).

## 3.1 Techniques of Differentiation

**Definition 3.2** (Product Rule)**.** If the functions $f$ and $g$ are differentiable, then

$$\frac{d}{dx}(f \cdot g) = g \cdot \frac{d}{dx} f + f \cdot \frac{d}{dx} g.$$

In Lagrange's notation (prime notation),

$$(f \cdot g)' = g \cdot f' + f \cdot g'.$$

**Definition 3.3** (Quotient Rule)**.** If the functions $f$ and $g$ are differentiable and $g \neq 0$, then

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g \cdot \frac{d}{dx} f - f \cdot \frac{d}{dx} g}{g^2}.$$

In Lagrange's notation (prime notation),

$$\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}.$$

**Definition 3.4** (Implicit Differentiation)**.** To differentiate implicit expressions, differentiate both sides of the equation then collect and isolate $dy/dx$.

**Definition 3.5** (Derivative of $a^x$)**.** The derivative of an exponential function is

$$\frac{d}{dx} a^x = a^x \cdot \ln a.$$

**Remark.** A proof of this is available in Appendix A.2.

**Definition 3.6** (Derivative of $e^x$). The derivative of the natural exponential function is

$$\frac{d}{dx}e^x = e^x \cdot \ln e = e^x.$$

**Remark.** An alternative proof of this is available in Appendix A.3.

**Definition 3.7** (Derivative of $\log_a x$). The derivative of a logarithmic function is

$$\frac{d}{dx}\log_a x = \frac{1}{x} \cdot \frac{1}{\ln a}.$$

**Remark.** A proof of this is available in Appendix A.4.

**Definition 3.8** (Derivative of $\ln x$). The derivative of the natural logarithm is

$$\frac{d}{dx}\ln x = \frac{d}{dx}\log_e x = \frac{1}{x} \cdot \frac{1}{\ln e} = \frac{1}{x}.$$

# 4 Applications of Differentiation

**Theorem 4.1** (Mean Value Theorem). If a function $f$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval $(a, b)$, then there exists at least one point $c \in (a, b)$ such that the instantaneous rate of change is equal to the average rate of change,

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

**Corollary 4.1** (Increasing and Decreasing Functions). Let a function $f$ be continuous on $[a, b]$ and differentiable on $(a, b)$. Then, if $f'(x) > 0$ ($f'(x) < 0$) for all $x \in (a, b)$, $f$ increases (decreases) on $[a, b]$.

**Remark.** A proof of this can be found in Appendix A.5.

**Theorem 4.2** (Concavity — The First Derivative). The differentiable function $f$ is concave up (down) on an interval $I$ if and only if $f'$ is increasing (decreasing) on $I$.

**Theorem 4.3** (Concavity — The Second Derivative). The twice-differentiable function $f$ is concave up (down) at $x$ if and only if $f''(x) > 0$ ($f''(x) < 0$).

**Remark.** A function is concave on an interval if the theorem holds for all $x$ in that interval.

## 4.1 Extrema

**Theorem 4.4** (Extreme Value Theorem)**.** If a function $f$ is continuous on a closed interval $[a, b]$, then $f$ attains both a maximum and minimum on that interval.

**Definition 4.1** (Critical Points)**.** For a function $f$, if $f'(c) = 0$ or $f'(c)$ does not exist, then $c$ is a critical point.

**Remark.** Endpoints of a closed interval are not considered critical points, but may still be extrema.

**Definition 4.2** (First Derivative Test)**.** Let the function $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$.

At a critical point $c$: if $f'(x) > 0$ ($f'(x) < 0$) for some $x < c$ and $f'(x) < 0$ ($f'(x) > 0$) for some $x > c$, then there is a local maximum (minimum) at $c$.

At a left endpoint $a$: if $f'(x) < 0$ ($f'(x) > 0$) for some $x > a$, then there is local maximum (minimum) at $a$.

At a right endpoint $b$: if $f'(x) > 0$ ($f'(x) < 0$) for some $x < b$, then there is local maximum (minimum) at $b$.

If the domain of a function is an open interval, there are no endpoints to classify as extrema.

**Definition 4.3** (Inflection Points)**.** On the graph of a function, $y = f(x)$, a point $(c, f(c))$ is a point of inflection if the concavity (the sign of $f''$) changes at $c$.

**Remark.** Inflection points can occur when $f''(x) = 0$ or $f''(x)$ does not exist.

**Definition 4.4** (Second Derivative Test)**.** Let the function $f$ be twice-differentiable at $c$. If $f'(c) = 0$ and $f''(c) < 0$ ($f''(c) > 0$), $f$ is concave down (up), then $f$ has a local maximum (minimum) at $c$.

**Remark.** If $f''(c) = 0$ or $f''(c)$ does not exist, then the Second Derivative Test fails and you must fall back to the First Derivative Test.

# 5 Integrals

**Theorem 5.1** (Fundamental Theorem of Calculus: Antiderivative)**.** If $f$ is continuous on $[a, b]$ then the function

$$F(x) = \int_a^x f(t)\,dt$$

is differentiable for all $x \in (a, b)$, and

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

**Definition 5.1** (Indefinite Integrals)**.** $F(x) = \int f(x)\,dx$ is a family of functions such that $\frac{d}{dx}F(x) = f(x)$.

**Theorem 5.2** (Fundamental Theorem of Calculus: Evaluation)**.** Let $f$ be a continuous function on $[a, b]$ then the function, and $F$ be a function such that $\frac{d}{dx}F(x) = f(x)$, then

$$\int_a^b f(t)\,dt = F(b) - F(a) = F(x)\Big|_a^b.$$

## 5.1 Properties of Integrals

**Theorem 5.3** (Max-Min Inequality)**.** Let $m$ be the minimum value of $f$ on $[a, b]$ and $M$ be the maximum value of $f$ on $[a, b]$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

**Theorem 5.4** (Domination)**.** If $f(x) \geq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

**Theorem 5.5** (Mean Value Theorem for Definite Integrals)**.** If $f$ is a continuous function on $[a, b]$, then there exists at least one point $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

In other words, the continuous function $f$ attains its average value at least once on $[a, b]$.

## 5.2 Techniques of Integration

**Theorem 5.6** (Integration by Substitution)**.** Let $u = g(x)$ be a continuous and differentiable function over an interval, let $f$ be continuous over the range of $g$ on that interval, and let $F$ be a function such that $\frac{d}{dx}F(x) = f(x)$. Then,

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du.$$

*Proof 5.1 (Integration by Substitution).* Let $u = g(x)$ be a continuous and differentiable function over an interval, and let $f$ be a continuous function over the range of $g$ on that interval.

Since

$$u = g(x) \implies du = g'(x)\, dx,$$

then

$$\int f(\underbrace{g(x)}_{u})\, \underbrace{g'(x)\, dx}_{du} = \int f(u)\, du.$$

■

**Theorem 5.7** (Integration By Parts)**.** Let $u$ and $v$ be continuous and differentiable functions, then

$$\begin{aligned} \int uv'\, dx &= \int (uv)'\, dx - \int vu'\, dx \\ &= uv - \int vu'\, dx. \end{aligned}$$

*Proof 5.2 (Integration by Parts).* Let $u$ and $v$ be continuous and differentiable functions. Then, by the Product Rule, we have

$$(uv)' = uv = vu' + uv' \implies uv' = (uv)' - vu'.$$

Integrating both sides with respect to $x$,

$$\begin{aligned}\int uv'\,dx &= \int (uv)'\,dx - \int vu'\,dx \\ &= uv - \int vu'\,dx.\end{aligned}$$

■

# 6 Applications of Integration

**Theorem 6.1** (Length of Curves)**.** Let $f$ be function that is continuous on $[a,b]$ and differentiable on $(a,b)$. Then, the length of the curve on $[a,b]$ is given by

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx.$$

**Remark.** See the proof/derivation in Appendix A.6.

## 6.1 Volume of Solids of Revolution

**Theorem 6.2** (Volume of Solids of Revolution: Disk/Washer Method)**.** The volume of the solid revolved on $[a,b]$ around an axis parallel to the $x$-axis is given by

$$V = \int_a^b \pi((R(x))^2 - (r(x))^2)\,dx,$$

where $R(x)$ is the outer radius and $r(x)$ is the inner radius of the washer.

**Remark.** The disk method is similar to the washer method, except the inner radius $r(x)$ is 0.

**Theorem 6.3** (Volume of Solids of Revolution: Cylindrical Shell Method). The volume of the solid revolved on $[a, b]$ around an axis perpendicular to the $x$-axis on is given by

$$V = \int_a^b 2\pi\, r(x)h(x)\, dx,$$

where $r(x)$ is the radius and $h(x)$ is the height of the cylinder.

Note that both the disk/washer and cylindrical shell methods can be performed and then integrated with respect to the $y$-axis, which may be helpful for some solids.

## 6.2 Improper Integrals

**Definition 6.1** (Improper Integrals: Infinity). If $f$ is continuous on $[a, \infty)$, then the improper integral $\int_a^\infty f(x)\, dx$ is defined as

$$\int_a^\infty f(x)\, dx = \lim_{t\to\infty} \int_a^t f(x)\, dx.$$

Similarly for $(-\infty, a]$, we define $\int_{-\infty}^a f(x)\, dx = \lim_{t\to-\infty} \int_t^a f(x)\, dx$.

If the limit exists, then we say that the integral converges, otherwise it diverges.

**Remark.** For a continuous function $f$ on $[a, \infty)$, if $\int_a^\infty f(x)\, dx$ converges, we must have $\lim_{x\to\infty} f(x) = 0$.

**Remark.** For a continuous function $f$ on $(-\infty, \infty)$, the improper integral $\int_{-\infty}^\infty f(x)\, dx$ means we consider 2 improper integrals

$$\int_{-\infty}^\infty f(x)\, dx = \int_{-\infty}^c f(x)\, dx + \int_c^\infty f(x)\, dx$$

for some constant $c$.

The integral $\int_{-\infty}^\infty f(x)\, dx$ converges only if both $\int_{-\infty}^c f(x)\, dx$ and $\int_c^\infty f(x)\, dx$ converge.

**Definition 6.2** (Improper Integrals: Undefined Point). If a function $f$ is continuous at every point of the interval $[a, b]$ except at $a$, then the improper integral is defined as

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx.$$

Similarly, if $f$ is continuous at every point of the interval $[a, b]$ except at $b$, then the improper integral is defined as

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx.$$

If the limit exists, then we say that the integral converges, otherwise it diverges.

**Remark.** If a function $f$ is continuous on $[a, b]$ except at a point $c \in (a, b)$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

# A Proofs

As I have not yet had formal education on proof writing, proofs in this document are not intended to be rigorous or advanced.

*Proof A.1 (l'Hôpital's Rule — Simple Version).* Let $c$ be an element of the real numbers system ($\mathbb{R}$), and let $f$ and $g$ be real-valued functions, differentiable at $c$.

Case: $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$
Since $f$ and $g$ are differentiable at $c$, they are also continuous at $c$. By definition, point continuity at $c$ implies that $\lim_{x\to c} f(x) = f(c) = 0$ and $\lim_{x\to c} g(x) = g(c) = 0$. So,

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{\frac{f(x)}{x-c}}{\frac{g(x)}{x-c}} = \lim_{x\to c} \frac{\frac{f(x)-0}{x-c}}{\frac{g(x)-0}{x-c}} = \lim_{x\to c} \frac{\frac{f(x)-f(c)}{x-c}}{\frac{g(x)-g(c)}{x-c}} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

■

*Proof A.2 (Derivative of $a^x$).* Let $a$ be a constant. If we let $y = a^x$ then $dy/dx$ is its derivative. Using Implicit Differentiation:

$$\begin{aligned}
\ln y &= \ln a^x \\
&= x \ln a \\
\frac{d}{dx} \ln y &= \frac{d}{dx} x \ln a \\
\frac{1}{y}\frac{dy}{dx} &= \ln a \\
\frac{dy}{dx} &= y \ln a \\
&= a^x \ln a.
\end{aligned}$$

■

*Proof A.3 (Derivative of $e^x$).* Using the Limit Definition of the Derivative, it is possible to determine the derivative of $e^x$.

$$\begin{aligned}\frac{d}{dx}e^x &= \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}\\ &= \lim_{h\to 0}\frac{e^x e^h - e^x}{h}\\ &= \lim_{h\to 0}\frac{e^x(e^h-1)}{h}\\ &= e^x\cdot\lim_{h\to 0}\frac{e^h-1}{h}\\ &= e^x\end{aligned}$$

$\lim_{h\to 0}\frac{e^h-1}{h} = 1$ can be determined graphically (or through l'Hôpital's Rule). ■

*Proof A.4 (Derivative of $\log_a x$).* Implicit Differentiation can be used to dermine the derivative of $y = \log_a x$.

$$\begin{aligned}y &= \log_a x\\ x &= a^y\\ \frac{d}{dx}x &= \frac{d}{dx}a^y\\ 1 &= a^y \ln a\cdot\frac{dy}{dx}\\ \frac{dy}{dx} &= \frac{1}{a^y}\cdot\frac{1}{\ln a}\\ &= \frac{1}{a^{\log_a x}}\cdot\frac{1}{\ln a}\\ &= \frac{1}{x}\cdot\frac{1}{\ln a}.\end{aligned}$$

■

*Proof A.5 (Increasing and Decreasing Functions).* Let $f$ be continuous on $[a,b]$ and differentiable on $(a,b)$, and let $x_1$ and $x_2$ be any two points on $[a,b]$ with $x_1 < x_2$. Applying the Mean Value Theorem to $f$ on $[x_1,x_2]$ gives

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

for some $c$ in $(x_1, x_2)$.

The left and right side of the equation must be equal and have the same sign. Since $x_1 < x_2 \implies x_2 - x_1 > 0$. This means the left side must have the same sign as $f'(c)$. We can conclude the following:

(a) if $f(x_2) > f(x_1)$, then $f(x_2) - f(x_1) > 0$ and $f'(c) > 0$, and

(b) if $f(x_2) < f(x_1)$, then $f(x_2) - f(x_1) < 0$ and $f'(c) < 0$.

So, if $f'(c) > 0$ for all $c$ in $(a, b)$, then $f(x_2) > f(x_1)$ (for all $x_2$ and $x_1$ in $[a, b]$ with $x_1 < x_2$), meaning $f$ is increasing on $[a, b]$. A similar procedure applies for $f'(c) < 0$. ■

*Proof A.6 (Length of Curves).* Let $y = f(x)$ be a continuous on $[a, b]$ and differentiable on $(a, b)$. Splitting the interval into $n$ subintervals of equal length, the length of the curve on the $i$th subinterval is given by

$$\begin{aligned}\Delta L_i &= \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \\ &= \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \cdot \Delta x_i.\end{aligned}$$

So, the length on the entire interval is

$$L \approx \sum_{i=1}^{n} \Delta L_i = \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \cdot \Delta x_i.$$

By the Mean Value Theorem, we have that there exists an $x_i^* \in (x_{i-1}, x_i)$ such that

$$f'(x_i^*) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = \frac{\Delta y_i}{\Delta x_i}.$$

Then, by the definition of integrals,

$$\begin{aligned}L &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + (f'(x_i^*))^2} \cdot \Delta x_i \\ &= \int_a^b \sqrt{1 + (f'(x))^2}\, dx.\end{aligned}$$

■